

Ginsparg-Wilson relation and the overlap formula

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Abstract

The fermionic determinant of a lattice Dirac operator that obeys the Ginsparg-Wilson relation factorizes into two factors that are complex conjugate of each other. Each factor is naturally associated with a single chiral fermion and can be realized as a overlap of two many body vacua.

The overlap formalism [1,2] provides a construction of lattice chiral gauge theories. The chiral determinant on a finite lattice is realized as an overlap of two many body vacua [1] and cannot be written as a path integral over a finite number of fermi fields on the lattice with a bilinear action. But the determinant of a massless Dirac fermion in the overlap formalism can be written as a path integral over a finite number of fermi fields with a bilinear action [3] and the lattice Dirac operator D obeys the Ginsparg-Wilson relation [4], namely [5]

$$\gamma_5 D + D \gamma_5 = D \gamma_5 D. \quad (1)$$

Recently, it was shown that any lattice fermion action

$$S_F = \sum_x \bar{\psi} D \psi \quad (2)$$

with D obeying the Ginsparg-Wilson relation (1) contains a continuous symmetry that can be viewed as the lattice chiral symmetry [6]. This would imply that the fermionic determinant, $\det D$, should factorize into two pieces with one being the complex conjugate of the other, if D obeys the Ginsparg-Wilson relation (1). Each piece would then be a single chiral determinant on the lattice associated with a left-handed or right-handed Weyl fermion. This is indeed the case for the particular form of D considered in [3] since the operator was obtained starting from the overlap formalism where the factorization into chiral pieces is built in. Here I show that the factorization holds as long as D obeys the Ginsparg-Wilson relation. The two factors are complex conjugates of each other. Further each factor can be realized as an overlap of two many body vacua. The proof of factorization presented here simply amounts to retracing the steps in [3].

Following Ref.[3,4], we define an operator \hat{H} through

$$D = 1 + \gamma_5 \hat{H}. \quad (3)$$

The Ginsparg-Wilson relation (1) reduces to

$$\hat{H}^2 = 1. \quad (4)$$

Therefore all the eigenvalues of \hat{H} are ± 1 . Starting in the chiral basis, let

$$U = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad (4)$$

be the unitary matrix that diagonalizes \hat{H} with

$$\hat{H} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & -\beta \\ \gamma & -\delta \end{pmatrix} \quad (5)$$

Under a rotation by U ,

$$U^\dagger D U = \begin{pmatrix} \alpha^\dagger & \gamma^\dagger \\ \beta^\dagger & \delta^\dagger \end{pmatrix} \begin{pmatrix} 2\alpha & 0 \\ 0 & 2\delta \end{pmatrix} \quad (6)$$

If \hat{H} has an equal number of positive and negative eigenvalues then α and δ are square matrices.[†] In this case,

$$\det D = \frac{\det \alpha^\dagger}{\det \delta} \det 2\alpha \det 2\delta = 2^{2N} \det \alpha \det \alpha^\dagger \quad (7)$$

where α and δ are assumed to be $N \times N$ matrices. The first factor on the right side of the first equality in (7) follows from the fact that U is an unitary matrix. Eqn. (7) is the factorization of D into two chiral factors one for each Weyl fermion. This factorization comes as no surprise since S_F in (2) contains a lattice chiral symmetry [6].

Since the factorization was obtained by simply retracing the steps in [3], $\det \alpha$ should be associated with an overlap formula. To see this, consider the two many body Hamiltonians,

$$\mathcal{H}^- = -a^\dagger \gamma_5 a; \quad \mathcal{H}^+ = -a^\dagger \hat{H} a \quad (8)$$

with a^\dagger and a are $2N$ fermion creation and annihilation operators obeying canonical anti-commutation relations. The matrix \hat{H} is the $2N \times 2N$ matrix in (3) and γ_5 is trivially extended to be a $2N \times 2N$ matrix. Let $|0\pm\rangle$ be the many body ground states of \mathcal{H}^\pm . The identity in Appendix B of Ref. [1] implies that

$$\langle 0- | 0+ \rangle = \det \alpha \quad (8)$$

with α being the submatrix of U in (4). Therefore each chiral factor in (7) is equal to an overlap of two many body vacua. Needless to say the phase of the many body ground

[†] If \hat{H} has an unequal number of positive and negative eigenvalues, α and δ are not square matrices and $\det D = 0$ implying that the background gauge field has a non-trivial topology.

states plays a crucial role in the proper construction of chiral gauge theories [1] but it does not affect vector gauge theories since $\det D$ is real and positive and independent of the phase of $\det \alpha$.

The specific form for \hat{H} in [3] is arrived at by writing $\hat{H} = \frac{H}{\sqrt{H^2}}$ with H being the Wilson realization of the continuum $\gamma_5[\gamma_\mu(\partial_\mu + iA_\mu(x) - m)]$ operator on the lattice for some $m > 0$. Any other discretization of the continuum operator can also be used. Eqn. (3) can be thought of as a method to construct a Dirac operator on the lattice with a lattice chiral symmetry by starting from some discretization of the continuum Dirac operator on the lattice that does not possess any chiral symmetry.

We have shown that the determinant of a lattice Dirac operator that obeys the Ginsparg-Wilson relation factorizes into pieces. The two pieces are complex conjugate of each other. Each piece is the determinant of a Weyl fermion and can be thought of as an overlap of two many body vacua.

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